

Imprints of energy limitation in transverse momentum distributions of jets

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Abstract. Using the Tsallis nonextensive approach, we analyse distributions of transverse spectra of jets. We discuss the possible influence of energy conservation laws on these distributions. Transverse spectra of jets exhibit a power-law behavior of $1/p_T^n$ with the power indices n similar to those for transverse spectra of hadrons.

1 Introduction

For some time now it has been known that transverse momentum spectra of different kinds measured in multi-particle production processes can be described by a quasi power-law formula¹,

$$h(p_T) = C \left(1 + \frac{p_T}{nT}\right)^{-n}, \quad (1)$$

which for large values of transverse momenta, $p_T \gg nT$, becomes scale-free (independent of T) power distribution, $1/p_T^n$. This was first proposed in [1,2] as the simplest formula extrapolating exponential behavior observed for low p_T to power behavior at large p_T . At present it is known as the QCD-inspired *Hagedorn formula* [3,4]. Distribution (1), nowadays, has been successfully used for the description of multiparticle production processes in a wide range of incident energy (from a few GeV up to a few of TeV) and in a broad range of transverse momenta (see, for example, reviews [5–7]). In particular, it turned out that it successfully describes transverse momenta of charged particles measured by the LHC experiments, the flux of which changes by over 14 orders of magnitude [8–11].

In many branches of physics eq. (1), with n replaced by $n = 1/(q-1)$, is widely known as the *Tsallis formula* [12–14]. In this form, eq. (1) is usually supposed to represent a nonextensive generalization of the Boltzmann-Gibbs exponential distribution, $\exp(-p_T/T)$, used in the statistical description of multiparticle production processes. Such

an approach is known as nonextensive statistics [12–14], in which the nonextensivity parameter q summarily describes all features causing a departure from the usual Boltzmann-Gibbs statistics. In particular it can be shown that the nonextensivity parameter q is directly related to the fluctuations of the parameter T identified with the “temperature” of the hadronizing fireball [15,16].

It was shown there that such a situation can occur when the heat bath is not homogeneous and must be described by a local temperature, T' , fluctuating from point to point around some equilibrium value, T . Assuming some simple diffusion picture as being responsible for equalization of this temperature [5,15,16] one obtains the evolution of T' in the form of a Langevin stochastic equation with the distribution of $1/T'$, $g(1/T')$, emerging as a solution of the corresponding Fokker-Planck equation. It turns out that in this case $g(1/T')$ takes the form of a gamma distribution,

$$g(1/T') = \frac{nT}{\Gamma(n)} \left(\frac{nT}{T'}\right)^{n-1} \exp\left(-\frac{nT}{T'}\right). \quad (2)$$

After convolution of the usual Boltzmann-Gibbs exponential factor $\exp(-p_T/T')$ with eq. (2) one immediately obtains a Tsallis distribution, eq. (1), with

$$n = \frac{\langle T' \rangle^2}{\text{Var}(T')}, \quad (3)$$

directly connected to the variance of T' . This idea was further developed in [19–21] (where problems connected with the notion of temperature in such cases were addressed). The above makes a basis for the so-called *superstatistics* [22,23].

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¹ Throughout the whole article natural units (Boltzmann constant and speed of light) are used by setting their values to unity.

Applications of Tsallis distributions to multiparticle production processes are now numerous. To those quoted previously in [5] one should add some new results from [17, 18, 6] and also presented in [21]. The most recent applications of this approach come from the STAR and PHENIX Collaborations at RHIC [24, 25] and from CMS [26, 27], ALICE [28–30] and ATLAS [31] Collaborations at LHC (see also a recent compilations [32, 33]).

It is now empirically well documented that transverse spectra of both hadrons and jets exhibit a power-law behavior of $1/p_T^n$ at high p_T . This observation (usually interpreted in terms of non-extensive Tsallis statistics) meets with difficulties. Whereas in [9, 10] it has been advocated that the power indices n for hadrons are systematically greater than those for jets, in [34] the values of corresponding power indices were found to be similar, strongly indicating the existence of a common mechanism behind all these processes.

In sect. 2 the influence of constraints, forcing the use of conditional probabilities for Tsallis distribution (and the case of Boltzmann statistics in appendix) is discussed. In sect. 3 we report the results concerning the consequences of the use of conditional Tsallis distribution for description of transverse spectra of jets. Section 4 is our summary and discussion of results.

2 Conditional probability — An influence of conservation laws

The influence of conservation laws on particle yields has been studied for a long time. In particular, particle momentum spectra within the Boltzmann-Gibbs approach are examined and deviations from the exponential spectrum of the grand canonical ensemble are quantified and discussed in [35, 36].

Deviations from the Tsallis distribution at large transverse momenta due to finite energy have been also discussed. Especially, hadron spectra in jets stemming from proton-proton and electron-positron collisions are discussed elsewhere [37, 38].

Let $\{E_{1,\dots,\nu}\}$ be a set of ν -independent identically distributed random variables described by some parameter T . For independent energies, $\{E_{i=1,\dots,\nu}\}$, each distributed according to the simple Boltzmann distribution,

$$g_1(E_i) = \frac{1}{T} \exp\left(-\frac{E_i}{T}\right), \quad (4)$$

the sum $U = \sum_{i=1}^{\nu} E_i$ is then distributed according to the gamma distribution,

$$g_{\nu}(U) = \frac{1}{T\Gamma(\nu)} \left(\frac{U}{T}\right)^{\nu-1} \exp\left(-\frac{U}{T}\right). \quad (5)$$

If the available energy U is limited, for example, if $U = \sum_{i=1}^{\nu} E_i = \text{const}$, then we have the following *conditional*

probability for the single-particle distribution, $g(E_i)$:

$$g(E_i|U) = \frac{g_1(E_i)g_{\nu-1}(U - E_i)}{g_{\nu}(U)} = \frac{(\nu-1)}{U} \left(1 - \frac{E_i}{U}\right)^{\nu-2}. \quad (6)$$

In the above consideration the value of the T parameter *does not fluctuate*. More details are given in the appendix and in the review [39].

Now let us consider a situation in which the parameter T in the joint probability distribution,

$$g(\{E_{1,\dots,\nu}\}) = \prod_{i=1}^{\nu} g_1(E_i), \quad (7)$$

fluctuates according to a gamma distribution, eq. (2). In this case we have a single-particle Tsallis distribution

$$h_{\nu}(E_i) = \frac{n-1}{nT} \left(1 + \frac{E_i}{nT}\right)^{-n} \quad (8)$$

and a distribution of $U = \sum_{i=1}^{\nu} E_i$ is given by (cf. [40])

$$h_{\nu}(U) = \frac{\Gamma(\nu+n-1)}{U\Gamma(\nu)\Gamma(n-1)} \left(\frac{U}{T}\right)^{\nu} \left(1 + \frac{U}{nT}\right)^{1-\nu-n}. \quad (9)$$

If the energy is limited, *i.e.*, if $U = \sum_{i=1}^{\nu} E_i = \text{const}$, we have the following conditional probability:

$$\begin{aligned} h(E_i|U) &= \frac{h_1(E_i)h_{\nu-1}(U - E_i)}{h_{\nu}(U)} \\ &= \frac{(\nu-1)(n-1)}{(n-2+\nu)} \frac{(nT+U)}{nTU} \left(\frac{U-E_i}{U}\right)^{\nu-1} \\ &\quad \times \left(1 + \frac{E_i}{nT}\right)^{-n} \left(1 - \frac{E_i}{nT+U}\right)^{2-\nu-n}. \end{aligned} \quad (10)$$

For $n \rightarrow \infty$, eq. (10) reduces to eq. (6). On the other hand, for large energy ($U \rightarrow \infty$) and large number of degrees of freedom ($\nu \rightarrow \infty$), the conditional probability distribution (10) reduces to the single-particle distribution given by eq. (8).

For $E_i \ll U$ the conditional probability (10) can be rewritten as

$$h(E_i|U) \simeq \frac{(\nu+2)(n-1)}{nT(n-2+\nu)} \left(1 + \frac{E_i}{nT}\right)^{-n}, \quad (11)$$

which, when additionally $\nu \gg 1$, reduces to eq. (8).

The results presented here are summarized in figs. 1 and 2 which show how large differences are between the *conditional* Tsallis distribution $h(E_i|U)$ and the *usual* $h_1(E_i)$.

3 Imprints of constraints in transverse spectra

Now we concentrate on the experimental distributions of transverse momenta,

$$f(p_T) = \frac{d^2\sigma}{p_T dp_T d\eta}, \quad (12)$$

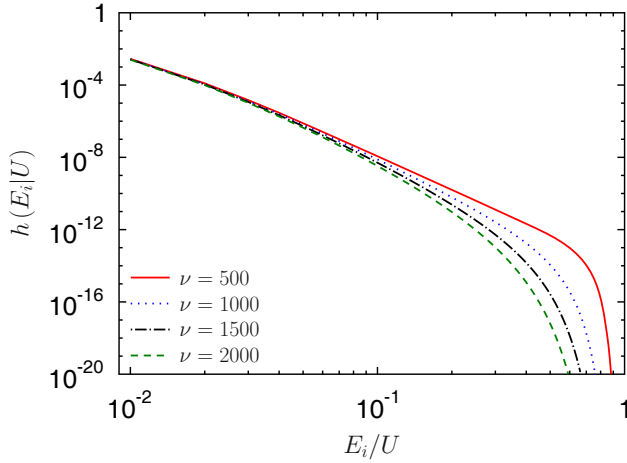


Fig. 1. (Color online) Conditional probability distribution, $h(E_i|U)$, as a function of E_i/U for Tsallis statistics ($n = 7$ and $T = 1$ GeV).

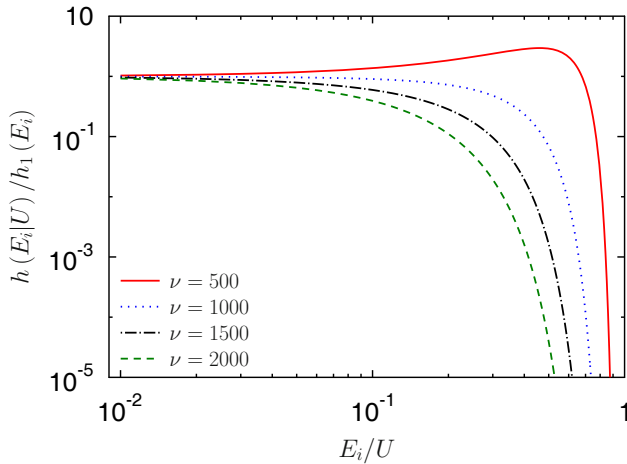


Fig. 2. (Color online) Ratio of the conditional distribution function $h(E_i|U)$ and the single-particle distribution $h_1(E_i)$ as a function of E_i/U , for Tsallis statistics ($n = 7$ and $T = 1$ GeV).

of jets observed at midrapidity and in the narrow jet cone defined by $R = \sqrt{\Delta\eta^2 + \Delta\phi^2}$ (where $\Delta\phi$ and $\Delta\eta$ are, respectively, the azimuthal angle and the pseudorapidity of hadrons relative to that of the jet). Data on $p+\bar{p}$ and $p+p$ interactions, covering a wide energy range from 0.54 TeV up to 7 TeV, obtained by CMS [41], CDF II [42], D0 [43] and UA2 [44] experiments were used for the analysis (cf. table 1 for more details). Because data, which we analyze, are presented at midrapidity (*i.e.*, for $\eta \sim 0$) and for large transverse momenta, $p_T \gg \mu$, the energy E of a produced jets is roughly equal to its transverse momentum p_T (and transverse energy E_T), which we shall use in what follows.

Transverse spectra for jets $f(p_T)$, fitted by the conditional Tsallis distribution $h(p_T|U)$ given by eq. (10), are shown in fig. 3. Parameters of the Tsallis distribution used

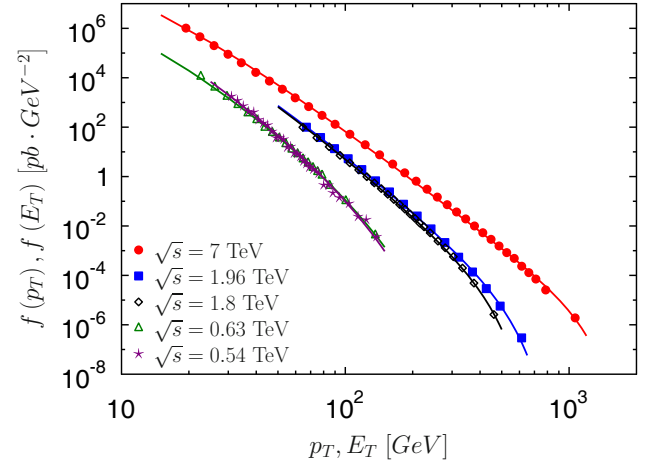


Fig. 3. (Color online) Transverse spectra for jets (p_T and E_T distribution are shown by full and open symbols, respectively) fitted by the conditional Tsallis distribution given by eq. (10).

Table 1. Jet production in $p + \bar{p}$ and $p + p$ interactions.

Exp.	Reaction	Variable	\sqrt{s} [TeV]	R	$ \eta $
CMS	$p + p$	p_T	7.0	0.5	< 0.5
CDF II	$p + \bar{p}$	p_T	1.96	0.7	< 0.1
D0	$p + \bar{p}$	E_T	1.8	0.7	< 0.5
D0	$p + \bar{p}$	E_T	0.63	0.7	< 0.5
UA2	$p + \bar{p}$	p_T	0.54	–	< 0.85

Table 2. Parameters of the Tsallis distribution used to fit the data.

Exp.	\sqrt{s} [TeV]	n	ν	T [GeV]	U [GeV]
CMS	7.0	6.95	1900	1.08	2200
CDF II	1.96	7.55	1340	1.08	1190
D0	1.8	7.7	1050	1.08	960
D0	0.63	8.0	800	1.08	500
UA2	0.54	8.3	800	1.08	490

to fit the data are shown in table 2. The available energy per degree of freedom,

$$\frac{U}{\nu} \cong 0.22 \cdot \ln \left(\frac{\sqrt{s}}{33} \right), \quad (13)$$

increases logarithmically with energy \sqrt{s} , while $\nu/\sqrt{s} \cong 12.7 - 2.73 \ln \sqrt{s} + 0.15 (\ln \sqrt{s})^2$ and $U/\sqrt{s} = 2.35 - 0.23 \ln \sqrt{s}$. The slope parameters decrease with energy as

$$n = 12.25 (\sqrt{s})^{-0.064} \quad (14)$$

and numerically are comparable with $n(s) = 1/(q(s) - 1)$ for $q(s)$ evaluated for transverse spectra of charged particles (for charge particles $n = 14.78(\sqrt{s})^{-0.09}$) [45].

Of course many other parametrisations than this given by eq. (14) are possible in our limited interval of energy (0.5 TeV–7 TeV). For example, $n = 4 + 10(\sqrt{s})^{-0.137}$ for jets and $n = 4 + 15.6(\sqrt{s})^{-0.2}$ for charged particles. However all parametrisations lead to the conclusion that $n_{\text{jet}} = n_{\text{char}}$ at TeV energies (1.2 TeV–1.4 TeV).

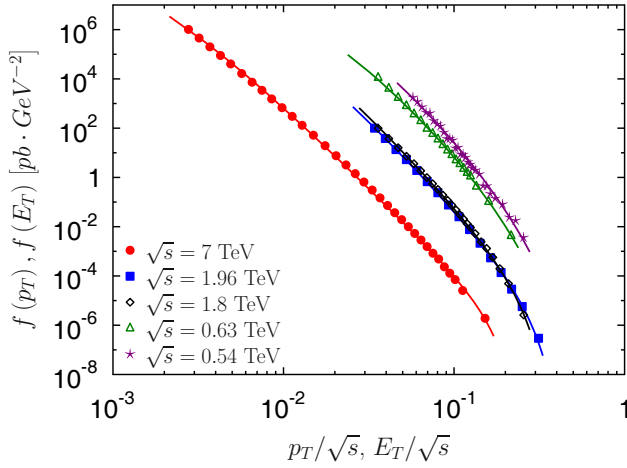


Fig. 4. (Color online) Jet differential yields as a function of p_T/\sqrt{s} or E_T/\sqrt{s} (p_T and E_T distributions are shown by full and open symbols, respectively) fitted by the conditional Tsallis distribution given by eq. (10).

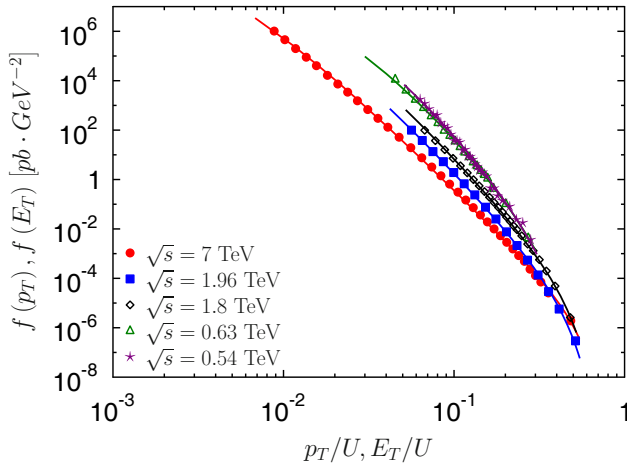


Fig. 5. (Color online) Jet differential yields as a function of p_T/U or E_T/U (p_T and E_T distribution are shown by full and open symbols, respectively) fitted by the conditional Tsallis distribution given by eq. (10).

The flux at given p_T (or E_T) increases with interaction energy \sqrt{s} as

$$f(p_T = 100 \text{ GeV}) = 5.5 \cdot 10^{-6} (\sqrt{s})^{1.85} - 0.6. \quad (15)$$

However a comparison of fluxes at transverse momenta close to the interaction energy (cf. fig. 4 shows a decrease with energy

$$f(p_T/\sqrt{s} = 0.1) = 10^{14} (\sqrt{s})^{-4.7}. \quad (16)$$

With increasing interaction energy, jets are produced with lower relative transverse momenta and paradoxically for lower \sqrt{s} we are closer to the kinematical limits. For transverse momenta close to available energy U , presented in fig. 5, fluxes of jets show rapid decrease with $p_T/U \rightarrow 1$.

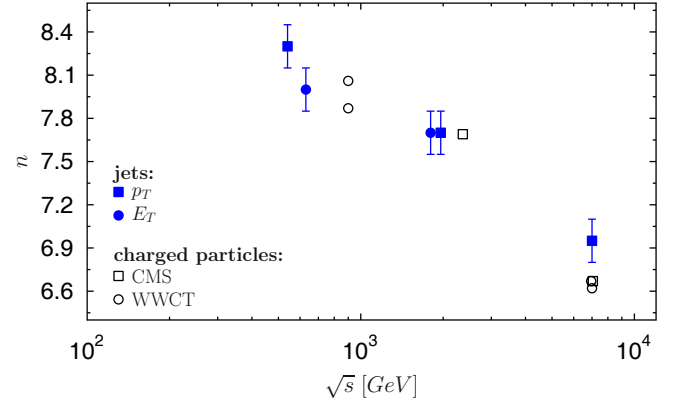


Fig. 6. (Color online) Slope parameters n extracted from the fits to the transverse spectra of jets (full symbols) in comparison with slope parameters for distributions of charged particles (open symbols: CMS [26,27] and WWCT [11]).

4 Summary

Transverse momentum spectra of jets, spanning over 12 orders of magnitude in the observed cross sections, can be successfully described by the conditional Tsallis distribution. Those spectra exhibit a power-law behavior of $1/p_T^n$ for which $n \approx 7-8$ ($n \approx 8$ at 800 GeV and decrease with energy, $n = 8(\sqrt{s}/800)^{-0.064}$). The observed deviation from the power law at tail of distribution is caused by kinematical constraints. Fluxes of jets for $p_T/\sqrt{s} = \text{const}$ are higher for lower incident energies, and are comparable for high p_T/U where fluxes have been lowering seemingly.

Comparison of power indices n for jets with those for charged particles produced in minimum bias collisions is shown in fig. 6. The values of the corresponding power indices are similar, indicating the existence of a common mechanism behind all these processes [34]. The suggestion that power indices for jets are almost 2 times smaller than those for hadrons [9,10] comes mainly from a different parametrisation of hadrons and jets spectra (not a simple Tsallis distribution) and cannot be accepted by looking directly at the corresponding distributions.

Appendix A. Conditional Boltzmann distribution

Actually, a conditional distribution given by eq. (6) emerges directly from the calculus of probability for the situation known as *induced partition* [46]. In short: $\nu - 1$ randomly chosen independent points $\{U_1, \dots, U_{\nu-1}\}$ breaks segment $(0, U)$ into ν parts, the length of which is distributed according to eq. (6). The length of the k^{th} such part corresponds to the value of energy $E_k = U_{k+1} - U_k$ (for ordered U_k). One could think of some analogy in physics to the case of random breaks of string in $\nu - 1$ points in the energy space.

We end this part by reminding how the Tsallis distribution with $q < 1$ arises from *statistical physics considerations*. Consider an isolated system with energy $U = \text{const}$

and with ν degrees of freedom. Choose a single degree of freedom with energy E (*i.e.* the remaining or reservoir energy is $E_r = U - E$). If this degree of freedom is in a single, well-defined state, then the number of states in the whole system equals $\Omega(U - E)$. The probability that the energy of the chosen degree of freedom equals E is then $P(E) \propto \Omega(U - E)$. Expanding (slowly varying) $\ln \Omega(E)$ around U ,

$$\ln \Omega(U - E) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\partial^{(k)} \ln \Omega}{\partial E_r^{(k)}}, \quad (\text{A.1})$$

and (because $E \ll U$) keeping only the two first terms, one gets $\ln P(E) \propto \ln \Omega(E) \propto -\beta E$, *i.e.*, $P(E)$ is a Boltzmann distribution given by eq. (4) with

$$\frac{1}{T} = \beta \stackrel{\text{def}}{=} \frac{\partial \ln \Omega(E_r)}{\partial E_r}. \quad (\text{A.2})$$

On the other hand, because one usually expects that $\Omega(E_r) \propto (E_r/\nu)^{\alpha_1 \nu - \alpha_2}$ (where $\alpha_{1,2}$ are of the order of unity and we put $\alpha_1 = 1$ and, to account for diminishing the number of states in the reservoir by one, $\alpha_2 = 2$) [47, 48], one can write

$$\frac{\partial^k \beta}{\partial E_r^k} \propto (-1)^k k! \frac{\nu - 2}{E_r^{k+1}} = (-1)^k k! \frac{\beta^{k-1}}{(\nu - 2)^k} \quad (\text{A.3})$$

and write the full series for probability of choosing energy E :

$$\begin{aligned} P(E) &\propto \frac{\Omega(U - E)}{\Omega(U)} = \exp \left[\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \frac{(-\beta E)^{k+1}}{(\nu - 2)^k} \right] \\ &= \beta^{\frac{\nu-1}{\nu-2}} \left(1 - \frac{1}{\nu-2} \beta E \right)^{(\nu-2)}, \end{aligned} \quad (\text{A.4})$$

where we have used the equality $\ln(1 + x) = \sum_{k=0}^{\infty} (-1)^k [x^{k+1}/(k+1)]$. This result, given by the Tsallis distribution with $q = (\nu - 3)/(\nu - 2) \leq 1$, coincides with the results from conditional probability and the induced partition.

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